Lesson 19

Residue Theorem

19.1 Residues

If $f(z)$ has a singularity at $z = z_0$ inside a simple closed curve C, but is otherwise analytic on C and inside C, then we can expand the function $f(z)$ in a Laurent series as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

This series is convergent for all points near $z = z_0$ (except at $z = z_0$) in the same domain of the form $0 < |z - z_0| < R$.

Now the coefficient b_1 of the first negative power $\frac{1}{z - z_0}$ of this Laurent series is given by

$$b_1 = \frac{1}{2\pi i} \int_C f(z^*)\, dz^*$$

$$\int_C f(z^*)\, dz^* = 2\pi i\, b_1 \qquad (19.1.1)$$

We define b_1 to be the residue of $f(z)$ at $z = z_0$ and denote it by

$$b_1$$

$$= \lim_{z = z_0} f(z) \qquad (19.1.2)$$

19.1.1 Examples:

1. We want to integrate $f(z) = z^{-4} \sin z$ around the unit circle C. Consider the Laurent series expansion as

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!\,z} + \frac{z}{5!} + \cdots$$

This is convergent for $|z| > 0$. Hence $b_1 = \frac{1}{3!}$.

2. Here we integrate $f(z) = \frac{1}{z^3 - z^4}$ clockwise around $|z| = \frac{1}{2}$. The function $f(z)$ has singularities at $z = 0$ and $z = 1$. However, $z = 1$ lies outside the circle C. So we can expand $f(z)$ in Laurent series at $z = 0$ as

$$f(z) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \cdots, \qquad 0 < |z| < 1$$

Note that the residue is 1 and we get

$$\int_C \frac{1}{z^3 - z^4}\,\mathrm{dz} \;=\; -2\pi i \lim_{z=0} Res\, f(z) \;=\; -2\pi i$$

19.1.2 Residue at Simple Pole

For a simple pole at $z = z_0$, the Laurent series is

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (0 < |z - z_0| < R).$$

This implies that

$$(z-z_0)f(z)=b_1+a_0(z-z_0)+a_1(z-\ z_0)^2+\cdots$$

So $\lim_{z=z_0}(z-z_0)f(z)=b_1=\lim_{z=z_0} Res\, f(z)$.

19.1.3 Example: $\lim_{z=i}\frac{q\ z+i}{z(z^2+1)}=\lim_{z=i}(z-i)\frac{q\ z+i}{z(z+i)(z-i)}=\frac{10i}{i(2i)}=-5i.$

19.1.3 Remark: Suppose we have $f(z)=\frac{p(z)}{q(z)}$, where p and q are analytic, $p(z_0)\neq 0$ and $q(z)$ has a simple pole z_0 so that $f(z)$ has a simple pole z_0. So by Taylor series, we find

$$q(z)=(z-\ z_0)q'(z_0)+\frac{(z-\ z_0)^2}{2!}q''(z_0)+\cdots$$

So $$\lim_{z=z_0}(z-z_0)f(z)=\lim_{z=z_0}(z-z_0)\frac{p(z)}{q(z)}$$

$$=\lim_{z=z_0}\frac{(z-z_0)p(z)}{(z-\ z_0)q'(z_0)+\frac{(z-\ z_0)^2}{2!}q''(z_0)+\cdots}$$

$$=\frac{p(z_0)}{q'(z_0)}$$

19.2 Residue at Pole of Any Order

If $f(z)$ has a pole of any order $m>1$ at $z=z_0$, then its Laurent series can be written as

$$f(z)=\frac{b_m}{(z-z_0)^m}+\frac{b_{m-1}}{(z-z_0)^{m-1}}+\cdots\frac{b_1}{z-z_0}+a_0+a_1(z-\ z_0)$$

where $b_m\neq 0$.

The residue of $f(z)$ at $z = z_0$ is b_1. If we multiply both sides by $(z - z_0)^m$, we get

$$(z - z_0)^m f(z) = b_m + b_{m-1}(z - z_0) + \cdots + b_1(z - z_0)^{m-1}$$
$$+a_0(z - z_0)^m + a_1(z - z_0)^{m+1} + \cdots$$

The residue b_1 of $f(z)$at $z = z_0$ is now the coefficient of the power $(z - z_0)^{m-1}$ in the Taylor series of the function $g(z) = (z - z_0)^{m-1} f(z)$ with center at $z = z_0$. So

$$b_1 = \frac{1}{(m-1)!} g^{(m-1)}(z_0)$$

(by Taylor's Theorem). Hence, if $f(z)$ has a pole of the m^{th}-order at $z = z_0$, the residue is given by

$$\lim_{z \to z_0} Res\, f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \right\}$$

19.2.1 Example: The function $f(z) = \frac{50z}{(z+4)(z-1)^2}$ has a pole of second order at $z = 1$. So

$$\lim_{z \to 1} Res\, f(z) = \lim_{z \to 1} \frac{d}{dz} \left[\frac{50z}{(z+4)} \right]$$

$$= \lim_{z \to 1} \frac{d}{dz} \left[\frac{50(z+4) - 50z}{(z+4)^2} \right] = \lim_{z \to 1} \frac{200}{(z+4)^4} = 8.$$

19.2.2 Residue Theorem: Let the function $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2 \dots, z_k$ inside C. Then the integral of $f(z)$ taken counter clockwise around C is given by

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

Proof: We enclose each of the singular points z_i in a circle C_i with radius small enough that these k circles and C are all separated. Then $f(z)$is analytic in the domain D bounded by C and $C_1, \dots, C_k$ and on the entire boundary of D. From Cauchy's integral theorem, we thus have

$$\int_C f(z)\,dz = \sum_{j=1}^{k} \int_{C_j} f(z)\,dz$$

$$= \sum_{j=1}^{k} 2\pi i \operatorname*{Res}_{z=z_j} f(z)$$

$$= 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z).$$

19.2.3 Examples:

1. Find $I = \oint_C \frac{4-3z}{z^2-z}\,dz$, where C is simple closed path that

(a) encloses 0 and 1,

(b) 0 is inside and 1 is outside,

(c) 0 and 1 are outside,

(d) 1 is inside, o is outside.

$$\operatorname*{Res}_{z=0}\frac{4-3z}{z(z-1)}=\lim_{z\to 0}\frac{4-3z}{z-1}=-4$$
$$\operatorname*{Res}_{z=1}\frac{4-3z}{z(z-1)}=\lim_{z\to 1}\frac{4-3z}{z}=1$$

Hence (a) $\boldsymbol{I=2\pi i\,(-4+1)=-6\pi i}$, (b) $\boldsymbol{I=-8\pi i}$, (c) 0 , (d) $\boldsymbol{2\pi i}$.

2. $\oint_C \frac{\tan z}{z^2-1}dz=2\pi i\left[\operatorname*{Res}_{z=1}\frac{\tan z}{z^2-1}+\operatorname*{Res}_{z=-1}\frac{\tan z}{z^2-1}\right]$

$$=2\pi i\left[\frac{\tan z}{2}+\frac{\tan(-1)}{-2}\right]=2\pi i\tan(1)$$

3. Evaluate $I=\oint_C\left(\frac{ze^{z\pi}}{z^4-16}+ze^{\frac{\pi}{z}}\right)dz$, C is ellipse $9x^2+y^2=9$.

The first term in the integrand has simple poles at $z=\pm 2$ and $z=\pm 2i$.

The poles at ± 2 lie outside the curve C. So the first pole of I is

$$I_1=2\pi i\left[\operatorname*{Res}_{z=2i}\frac{ze^{z\pi}}{z^4-16}+\operatorname*{Res}_{z=-2i}\frac{ze^{z\pi}}{z^4-16}\right]$$
$$=2\pi i\left[\lim_{z\to 2i}\frac{ze^{z\pi}}{(z^2-4)(z+2i)}+\lim_{z\to -2i}\frac{ze^{z\pi}}{(z^2-4)(z+2i)}\right]$$
$$=2\pi i\left[\frac{2i}{-32\,i}+\frac{-2i}{32\,i}\right]=\frac{-\pi i}{4}$$

For the second term

$$ze^{\frac{\pi}{z}} = z\left[1 + \frac{\pi}{z} + \frac{\pi^2}{2!\,z^2} + \cdots\right]$$

$$= z + \pi + \frac{\pi^2}{2z} + \cdots$$

So $\operatorname*{\mathbf{Res}}_{z=0} ze^{\frac{\pi}{z}} = \frac{\pi^2}{2}$, and then the second term of the integral is $2\pi i\,\frac{\pi^2}{2} = \pi^3 i$.

Hence $I = \pi^3 i - \frac{\pi i}{4} = \pi i\,(\pi^2 - \frac{1}{4})$.

4. Evaluate the integral $\oint_C \frac{z-23}{z^4-4z-5}dz$, $\quad C: |z-2| = 4$. We can write

$$\oint_C \frac{z-23}{z^4-4z-5}dz = \oint_C \frac{z-23}{(z-5)(z+1)}dz$$

$$= 2\pi i\left[\operatorname*{Res}_{z=5}\frac{z-23}{(z-5)(z+1)} + \operatorname*{Res}_{z=-1}\frac{z-23}{(z-5)(z+1)}\right]$$

$$= 2\pi i\left[\frac{-18}{6} + \frac{-24}{-6}\right] = 2\pi i$$

5. Evaluate $I = \int_{|z|=1} \tan(\pi z)\,\mathrm{d}z$. The function $\tan(\pi z)$ has simple poles at $z = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$ of which only $z = \pm\frac{1}{2}$ lie inside the contour. So

$$I = 2\pi i\left[\operatorname*{Res}_{z=\frac{1}{2}}\tan(\pi z) + \operatorname*{Res}_{z=-\frac{1}{2}}\tan(\pi z)\right]$$

$$= 2\pi i\left[\lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\tan(\pi z) + \lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)\tan(\pi z)\right]$$

$$= 2\pi i\left[-\frac{1}{\pi} - \frac{1}{\pi}\right] = -4i\,.$$

6. Evaluate $\oint_C \frac{\cosh(\pi z)}{z^4+5z^2+4} dz, \quad C: |z| = 4.$

The integral has simple poles at $z = \pm i, \pm 2i$ and they all lie inside the contour.

Now for pole at $z = a$

$$\operatorname*{Res}_{z=a} f(z) = \frac{\phi(a)}{\psi'(a)} = \frac{a\cosh(a\pi)}{4\,a^3+10a} = \frac{\cosh(a\pi)}{4\,a^2+10}.$$

So,

$$I = 2\pi i\left[\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) + \operatorname*{Res}_{z=2i} f(z) + \operatorname*{Res}_{z=-2i} f(z)\right]$$

$$= 2\pi i\left[\frac{\cosh(\pi i)}{6} + \frac{\cosh(\pi i)}{6} - \frac{\cosh(2\pi i)}{6} - \frac{\cosh(2\pi i)}{6}\right] = -\frac{4\pi i}{3}$$

Suggested Readings

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